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Birkhoff–Gustavson normal form and the semiclassical energies of a double pendulum

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Abstract. Although the double pendulum is a well known system, the literature is sparse with regard to the study of its full dynamical behaviour. As a simple conservative system that displays both ordered and chaotic motion, it poses an interesting problem for the study of classical–quantum correspondence. In this work, we present its Birkhoff–Gustavson normal form and semiclassical energies obtained from the normal form. The Padé summation technique is used to re-sum the normal-form series.

1. Introduction

The familiar system of a double pendulum has gained some new attention in dealing with the manifestation of chaos in classical and quantum systems [1]. This system has a number of appealing features. For example, its dynamics has the full richness of classical nonlinear time evolution and yet it does not require an elaborate experimental setup [2] to observe this complex behaviour. Its observable chaotic motion has raised some questions that quantum mechanics does not seem to have clear answers for [1]. We feel that it is worthwhile to have a closer look at this system. As for its semiclassical and quantum mechanical treatments, the literature is almost non-existent. The problem with a quantum calculation of the double pendulum arises partly from the fact that there is no method known (to the authors) for its exact quantization. The momenta and coordinates remain coupled in a way that hinders unambiguous quantization. In this work, we present our results of the Birkhoff–Gustavson normal form [3] analysis of the system. This normal-form approach provides valuable classical, semiclassical and quantum mechanical information about dynamical systems near their equilibrium points [4]. A reliable quantization of the system should account for the limiting results obtainable from the normal form.

2. The dynamical system

We consider an idealized planar double pendulum with point masses m_1 and m_2 attached to two inextensible massless arms of lengths l_1 and l_2. The system has two degrees of freedom and its Lagrangian in polar coordinates θ_1 and θ_2 (figure 1) takes the form

$$\mathcal{L} = \mathcal{T} - \mathcal{V}$$

$$\mathcal{T} = \frac{(m_1 + m_2)\, l_1{}^2 \dot{\theta}_1^2}{2} + \frac{m_2\, l_2{}^2 \dot{\theta}_2^2}{2} + m_2\, l_1\, l_2\, \dot{\theta}_1\, \dot{\theta}_2\, \cos(\theta_1 - \theta_2)$$

$$\mathcal{V} = (m_1 + m_2)\; g l_1\; (1 - \cos\theta_1) + m_2\, g l_2\; (1 - \cos\theta_2)$$

$$\tau = \text{time}.$$

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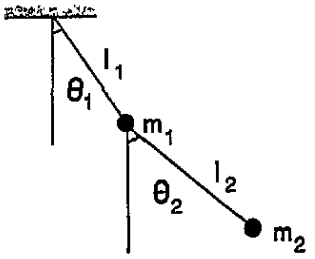

Figure 1. The double pendulum.

Note that the time derivatives $\dot{\theta}_1$ and $\dot{\theta}_2$ are taken with respect to time τ. It is convenient to use dimensionless quantities, and for this purpose, we scale the time τ as

$$\tau = \sqrt{m_1 l_1}\, t/\sqrt{\alpha}\,.$$

This scaling gives a new dimensionless Lagrangian $L = \mathcal{L}/\alpha$, where α has the dimension of energy. It is clear that L and $\mathcal{L}$ describe the same dynamics and we have

$$\begin{aligned} L &= T - V \\ T &= \frac{(1+m)\dot{\theta}_1{}^2}{2} + \frac{ml^2\dot{\theta}_2^2}{2} + ml\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \\ V &= \gamma\left[(1+m)(1-\cos\theta_1) + ml(1-\cos\theta_2)\right]. \end{aligned} \tag{1}$$

Here $m = m_2/m_1$, $l = l_2/l_1$, $\gamma = m_1 l_1 g/\alpha$, and g is the acceleration due to gravity. Without any loss of generality, we will use $\gamma = 1$, which means that the total energy will be measured in units of $m_1 l_1 g$. Also, we note here, for the purposes of what is to follow, that the above corresponds to the scaling of $\hbar$ as $\hbar \to \sqrt{m_1\alpha l_1}\,\hbar$ with the frequencies of the uncoupled zeroth-order Hamiltonians scaling as $\omega \to \sqrt{g/l_1}\,\omega$. The corresponding dimensionless classical Hamiltonian of the system is given by

$$\begin{aligned} H = {} & \frac{P_1^2/2 + (1+m)P_2^2/(2ml^2) - P_1P_2\cos(\theta_1-\theta_2)/l}{1+m\,\sin^2(\theta_1-\theta_2)} \\ & +\gamma\left\{(1+m)(1-\cos\theta_1) + m\,l(1-\cos\theta_2)\right\}. \end{aligned} \tag{2}$$

For the normal-form analysis, we expand H about the equilibrium point $\theta_1 = 0$, $\theta_2 = 0$ in the following manner:

$$H = \sum_{n=0} \frac{\epsilon^n H_n}{n!} \tag{3}$$

where ϵ is the perturbation expansion parameter (here we use $\epsilon = 1$). The expansion of H is straightforward [5] and the first few terms are given below:

$$\begin{aligned} H_0 &= \frac{P_1{}^2}{2} + \frac{(1+m)\,P_2{}^2}{2ml^2} - \frac{P_1\,P_2}{l} + \frac{(1+m)\,\theta_1{}^2}{2} + \frac{ml\theta_2{}^2}{2} \\ H_1 &= -\frac{(1+m)\,\theta_1{}^4}{24} - \frac{(1+m)\,\theta_1{}^2P_2{}^2}{2l^2} + \frac{(1+2m)\,\theta_1{}^2P_1\,P_2}{2l} - \frac{\theta_1{}^2mP_1{}^2}{2} \\ &\quad + \frac{(1+m)\,\theta_1\,\theta_2\,P_2{}^2}{l^2} - \frac{(1+2m)\,\theta_1\,\theta_2\,P_1\,P_2}{l} + \theta_2\,\theta_1\,mP_1{}^2 - \frac{(1+m)\,\theta_2{}^2P_2{}^2}{2l^2} \\ &\quad + \frac{(1+2m)\,\theta_2{}^2P_1\,P_2}{2l} - \frac{\theta_2{}^2mP_1{}^2}{2} - \frac{\theta_2{}^4ml}{24} \end{aligned} \tag{4}$$

$$H_2 = \frac{m+1}{360}\theta_1^6 + \frac{(1+m)(3m+1)\theta_1^4 P_2^2}{3l^2} - \frac{(1+20m+24m^2)\theta_1^4 P_1 P_2}{12l}$$
$$+\frac{m(3m+1)\theta_1^4 P_1^2}{3} - \frac{(4+4m)(3m+1)\theta_1^3\theta_2 P_2^2}{3l^2}$$
$$+\frac{(1+20m+24m^2)\theta_1^3 P_1\theta_2 P_2}{3l} - \frac{4m(3m+1)\theta_1^3 P_1^2\theta_2}{3}$$
$$+\frac{(2+2m)(3m+1)\theta_1^2\theta_2^2 P_2^2}{l^2} - \frac{(1+20m+24m^2)\theta_1^2 P_1\theta_2^2 P_2}{2l}$$
$$+2m(3m+1)\theta_1^2 P_1^2\theta_2^2 - \frac{(4+4m)(3m+1)\theta_1\theta_2^3 P_2^2}{3l^2}$$
$$+\frac{(1+20m+24m^2)\theta_1 P_1\theta_2^3 P_2}{3l} - \frac{4m(3m+1)\theta_1 P_1^2\theta_2^3}{3}$$
$$+\frac{(1+m)(3m+1)\theta_2^4 P_2^2}{3l^2} - \frac{(1+20m+24m^2)P_1\theta_2^4 P_2}{12l}$$
$$+\frac{m(3m+1)P_1^2\theta_2^4}{3} + \frac{ml\theta_2^6}{360}.$$

The above series in θ_1 and θ_2 for H does not converge if, in (2), the term $m\sin^2(\theta_1-\theta_2) \geqslant 1$. This problem will affect the convergence of the normal-form series as discussed in section 3. The zeroth-order Hamiltonian H_0 is decoupled by the following canonical transformation:

$$\theta_1 = a_1\mathcal{X}_1 + a_2\mathcal{X}_2 \qquad \theta_2 = a_3\mathcal{X}_1 + a_4\mathcal{X}_2$$
$$P_1 = \{a_1(1+m) + mla_3\}\omega_1\mathcal{Z}_1 + \{a_2(1+m) + mla_4\}\omega_2\mathcal{Z}_2$$
$$P_2 = ml(a_1 + la_3)\omega_1\mathcal{Z}_1 + ml(a_2 + la_4)\omega_2\mathcal{Z}_2$$
$$a_1 = -\frac{\sqrt{m}\sqrt{l}\omega_1^{3/2}}{\sqrt{m+1}\sqrt[4]{q}\sqrt{\omega_1^2-1}} \qquad a_2 = \frac{\sqrt{m}\sqrt{l}\omega_2^{3/2}}{\sqrt{m+1}\sqrt[4]{q}\sqrt{1-\omega_2^2}} \tag{5}$$
$$a_3 = \frac{\sqrt{m+1}\sqrt{\omega_1^2-1}}{\sqrt{m}\sqrt{l}\sqrt[4]{q}\sqrt{\omega_1}} \qquad a_4 = \frac{\sqrt{1+m}\sqrt{1-\omega_2^2}}{\sqrt{m}\sqrt{l}\sqrt[4]{q}\sqrt{\omega_2}}$$
$$\omega_1 = \frac{\sqrt{m+1+lm+l+\sqrt{q}\sqrt{2}}}{2\sqrt{l}} \qquad \omega_2 = \frac{\sqrt{m+lm+l+1-\sqrt{q}\sqrt{2}}}{2\sqrt{l}}$$
$$q = l^2 + 2ml^2 - 2l + m^2l^2 + 2m^2l + 1 + 2m + m^2.$$

The decoupled zeroth-order Hamiltonian then becomes equal to the sum of two independent harmonic oscillator Hamiltonians, and is written as

$$H_0 = \tau_1 + \tau_2 \qquad \tau_1 = \frac{\omega_1}{2}(\mathcal{Z}_1^2 + \mathcal{X}_1^2) \qquad \tau_2 = \frac{\omega_2}{2}(\mathcal{Z}_2^2 + \mathcal{X}_2^2). \tag{6}$$

For the normal-form analysis in the following section, all the perturbation expansion terms H_n are expressed in terms of the variables $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$ which, in turn, are transformed to normal-form coordinates (X, Z) in section 3 by using (7).

3. Birkhoff–Gustavson normal form

The Birkhoff–Gustavson normal form is a canonical perturbation technique [6]. It is the classical analogue [7, 8] of the quantum mechanical Rayleigh–Schrödinger perturbation method, and it has been used in the past [4, 9–11] with considerable success in generating constants of motion and semiclassical energies of nonlinear systems. Since the normal-form

technique can provide valuable insight into the dynamics of systems near their equilibrium points, we employ it to study the double pendulum. The details of the method are available in the literature [7]. We present a brief description for the sake of completeness. For the Lie transform used in generating the normal-form series K, we define a generating function W as

$$W = \sum_{n=0} \frac{\epsilon^n W_{n+1}}{n!}$$

which yields the recursion relation

$$\begin{aligned} f_n^k &= f_{n+1}^{k-1} + \sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!} L_{m+1} f_{n-m}^{k-1} \qquad k \geqslant 1,\ m \geqslant 0 \\ L_j f &= \{f, W_j\} . \end{aligned} \tag{7}$$

Here $\{A, B\}$ signifies the Poisson bracket of A and B. By defining $f^n{}_0 = K_n$ and $f^0{}_n = H_n$, we obtain from the recursion relation the normal-form series as

$$K = \sum_{n=0} \frac{\epsilon^n K_n}{n!} . \tag{8}$$

The first few terms of the series are given explicitly as

$$\begin{aligned} K_0 &= H_0 \\ K_1 &= H_1 + \{H_0, W_1\} \\ K_2 &= H_2 + 2\{H_1, W_1\} + \{H_0, W_2\} + \{\{H_0, W_1\}, W_1\} . \end{aligned} \tag{9}$$

For convenience in determining the K_n, we introduce the new coordinates (ξ, η) such that the normal-form coordinates (X, Z) become

$$X_j = \frac{\xi_j + \iota\,\eta_j}{\sqrt{2}} \qquad Z_j = \frac{\iota\,\xi_j + \eta_j}{\sqrt{2}} \qquad j = 1, 2 \qquad \iota = \sqrt{-1}$$

and defining h_1 and h_2 as

$$h_1 = \frac{\omega_1}{2}(Z_1^2 + X_1^2) \qquad h_2 = \frac{\omega_2}{2}(Z_2^2 + X_2^2)$$

we have

$$K_0 = h_1 + h_2 .$$

In terms of ξ_1, ξ_2, η_1, η_2, the H_n , K_n and W_{n-1} are homogeneous polynomials of degree $2(n+1)$. Our task now is to determine the W_n and make sure that all the normal-form terms K_n satisfy the relation $\{K_0, K_n\} = 0$. This task becomes remarkably simple if the above coordinate transformation is used, as can be seen from the following discussion.

With the ξ , η coordinates, we have

$$K_0 = \iota\,\omega_1\,\xi_1\,\eta_1 + \iota\,\omega_2\,\xi_2\,\eta_2$$

and, for any function $f(\xi_1, \xi_2, \eta_1, \eta_2)$, the Poisson bracket with K_0 is given by

$$\{K_0, f\} = \iota(\omega_1 D_1 + \omega_2 D_2) f$$

where

$$D_j = \eta_j \frac{\partial}{\partial \eta_j} - \xi_j \frac{\partial}{\partial \xi_j} \qquad j = 1, 2 .$$

Thus a monomial $\xi_1^{\mu}\xi_2^{\nu}\eta_1^{\rho}\eta_2^{\sigma}$ is a simultaneous eigenfunction of the operators D_1 and D_2 with eigenvalues $\rho - \mu$ and $\sigma - \nu$, respectively. It is therefore simple to decide whether

such a monomial belongs to the normal form. For this to be the case, the monomial must satisfy the relation $\omega_1(\rho-\mu)+\omega_2(\sigma-\nu)=0$. This is a definite computational advantage.

For arbitrary l and m, the K_n can be computed but the higher-order terms become too complicated to present in a short article. We report here only K_0 and K_1 for arbitrary l and m. We also report K through K_8 for the case $m=4$, $l=\frac{5}{4}$ as an example. The resonant and non-resonant cases behave differently and they are reported below under separate headings. We say that there is an $n:m$ resonance if there exist integers n and m such that $n\omega_1+m\omega_2=0$.

3.1. Non-resonant normal form

For arbitrary l and m, the first two terms of the normal-form expansion for the non-resonant case are obtained from (9) and are given as follows:

$$
\begin{aligned}
K_0 &= h_1+h_2\\
K_1 &= \left(\frac{la_1\,ma_3\,(a_1-a_3)^2}{4}-\frac{mla_3{}^4+(m+1)\,a_1{}^4}{16\,\omega_1{}^2}\right)h_1{}^2\\
&\quad+\left(\frac{la_1\,ma_3\,(a_4-a_2)^2\,\omega_1}{2\,\omega_2}+\frac{lma_4\,a_2\,(a_1-a_3)^2\,\omega_2}{2\,\omega_1}\right)h_1\,h_2\\
&\quad-\left(\frac{mla_3{}^2a_4{}^2+(m+1)\,a_1{}^2a_2{}^2}{4\,\omega_1\,\omega_2}\right)h_1\,h_2+\left(\frac{lma_4\,a_2\,(a_4-a_2)^2}{4}\right)h_2{}^2\\
&\quad-\left(\frac{mla_4{}^4+(m+1)\,a_2{}^4}{16\,\omega_2{}^2}\right)h_2{}^2.
\end{aligned}
\tag{10}
$$

In (10), K_0 and K_1 are polynomials in h_1 and h_2. For all non-resonant cases, the K_n are polynomials in h_1 and h_2. However, this is not so in resonant cases, as discussed below.

3.2. Resonant normal form

For the double pendulum we have $1\leqslant\omega_1$ and $0\leqslant\omega_2\leqslant 1$. An example of a 2:1 resonance is the case when $l=1$, $m=\frac{9}{16}$. Similarly a 3:1 resonance can be obtained, for instance, by setting $l=1$, $m=\frac{16}{9}$.

3.2.1. 2:1 Resonance ($\omega_1=2\,\omega_2$, $\omega_2=\omega$). The K_0 and K_1 terms are obtained from (10) by substituting $\omega_1=2\omega$ and $\omega_2=\omega$. Both K_0 and K_1 are polynomials in h_1 and h_2. However, higher-order normal-form terms for 2:1 resonances cannot be expressed as polynomials in h_1 and h_2 (see 3:1 resonance below).

3.2.2. 3:1 Resonance ($\omega_1=3\,\omega_2$, $\omega_2=\omega$). Unlike the non-resonant and 2:1 resonant cases, the K_1 term (as well as higher-order terms) for the 3:1 resonant case cannot be expressed as a polynomial in h_1 and h_2. We present here only K_1:

$$K_1\,[\text{resonant}]=K_1[\text{non-resonant}]+(T_1+T_2)\,T_3 \tag{11}$$

where

$$T_1=\frac{(a_4-a_2)\left(3\,a_1\,a_4{}^2-a_4\,a_1\,a_2+a_4\,a_2\,a_3-3\,a_2{}^2a_3\right)\omega^2ml}{16}$$

$$T_2 = -\frac{(1+m)\,a_1\,a_2{}^3}{48} - \frac{m l a_3\,a_4{}^3}{48}$$

$$T_3 = \left(X_2{}^2 - 3\,Z_2{}^2\right) X_1\,X_2 + \left(3\,X_2{}^2 - Z_2{}^2\right) Z_1\,Z_2\,.$$

The K_0 and K_1 (non-resonant) terms are obtained from (10) by substituting $\omega_1 = 3\,\omega$ and $\omega_2 = \omega$.

An observation is in order here. If one substitutes $x_1 = \sin\theta_1$ and $x_2 = \sin\theta_2$ into the Lagrangian given in (1), then expands the corresponding Hamiltonian, one obtains a perturbation series in x_1, x_2 and their conjugate momenta that agrees only in the zeroth-order term with the perturbation series of H given in section 2. This is so in the sense that the functional form of H_0 in (4) is exactly the same in either of the two sets of phase-space variables. The higher-order terms, on the other hand, differ for the two choices of coordinates. Thus, it is not obvious that the normal form resulting from such an alternate expansion will agree with that presented in this work. It is, however, curiously interesting to note that all the non-resonant normal forms that we have studied in this work are the same for both (x_1, x_2, their conjugate momenta) and ($\theta_1, \theta_2, P_1, P_2$) expansions of H.

3.2.3. Normal form for the case $m = 4$, $l = \frac{5}{4}$. In order to discuss the normal-form analysis in more detail, we consider the case $m = 4$, $l = \frac{5}{4}$ as an example. This is a non-resonant case and the series [5] through K_8 is given below:

$$\omega_1 = 0.292\,080\,962\,6 \times 10^1 \qquad \omega_2 = 0.684\,741\,649 \times 10^0$$

$$\begin{aligned}
K = {} & h_1 + h_2 - 0.427\,974\,433\,4 \times 10^0\,h_1^2 + 0.184\,615\,384\,6 \times 10^{-1}\,h_1\,h_2 \\
& -0.625\,633\,581\,4 \times 10^{-2}\,h_2^2 + 0.549\,964\,363\,2 \times 10^0\,h_1^3 \\
& +0.101\,505\,847\,6 \times 10^{-1}\,h_2\,h_1^2 - 0.214\,750\,568\,3 \times 10^{-2}\,h_2^2\,h_1 \\
& -0.557\,988\,194\,9 \times 10^{-4}\,h_2^3 - 0.981\,362\,301\,9 \times 10^0\,h_1^4 \\
& -0.359\,639\,009 \times 10^{-1}\,h_1^3\,h_2 - 0.522\,101\,284\,6 \times 10^{-3}\,h_2^2\,h_1^2 \\
& -0.130\,589\,628 \times 10^{-4}\,h_2^3\,h_1 + 0.105\,396\,445\,9 \times 10^{-5}\,h_2^4 \\
& +0.207\,612\,839\,3 \times 10^1\,h_1^5 + 0.114\,734\,190\,6 \times 10^0\,h_2\,h_1^4 \\
& +0.332\,949\,806\,6 \times 10^{-2}\,h_2^2\,h_1^3 - 0.573\,573\,430\,2 \times 10^{-4}\,h_2^3\,h_1^2 \\
& +0.716\,633\,947\,3 \times 10^{-5}\,h_1\,h_2^4 + 0.343\,578\,489\,2 \times 10^{-8}\,h_2^5 \\
& -0.488\,821\,264\,4 \times 10^1\,h_1^6 - 0.367\,441\,241\,6 \times 10^0\,h_1^5\,h_2 \\
& -0.134\,453\,072\,8 \times 10^{-1}\,h_1^4\,h_2^2 - 0.124\,145\,445\,6 \times 10^{-2}\,h_2^3\,h_1^3 \\
& +0.684\,752\,581\,5 \times 10^{-4}\,h_2^4\,h_1^2 - 0.350\,538\,336\,3 \times 10^{-5}\,h_2^5\,h_1 \\
& +0.411\,754\,691\,4 \times 10^{-7}\,h_2^6 + 0.124\,012\,742\,4 \times 10^2\,h_1^7 \\
& +0.120\,302\,779\,4 \times 10^1\,h_2\,h_1^6 + 0.538\,345\,080\,4 \times 10^{-1}\,h_1^5\,h_2^2 \\
& +0.234\,674\,341\,2 \times 10^{-2}\,h_2^3\,h_1^4 + 0.362\,661\,710\,4 \times 10^{-3}\,h_1^3\,h_2^4 \\
& -0.566\,809\,161\,5 \times 10^{-5}\,h_1^2\,h_2^5 - 0.390\,208\,514\,6 \times 10^{-6}\,h_2^6\,h_1 \\
& +0.348\,779\,871\,5 \times 10^{-8}\,h_2^7 - 0.332\,610\,224 \times 10^2\,h_1^8 \\
& -0.402\,779\,026\,6 \times 10^1\,h_2\,h_1^7 - 0.205\,218\,359\,1 \times 10^0\,h_2^2\,h_1^6 \\
& -0.151\,286\,907\,4 \times 10^{-1}\,h_2^3\,h_1^5 + 0.257\,871\,5565 \times 10^{-3}\,h_1^4\,h_2^4 \\
& -0.246\,487\,035\,5 \times 10^{-3}\,h_1^3\,h_2^5 + 0.110\,929\,283\,6 \times 10^{-4}\,h_1^2\,h_2^6 \\
& -0.983\,148\,4723 \times 10^{-7}\,h_1\,h_2^7 + 0.313\,764\,076\,7 \times 10^{-9}\,h_2^8 \\
& +0.931\,596\,593\,2 \times 10^2\,h_1^9 + 0.137\,539\,928\,9 \times 10^2\,h_1^8\,h_2
\end{aligned}$$

$$+0.790\,573\,409\,7 \times 10^{0}\, h_2^2 h_1^7 + 0.538\,537\,719\,5 \times 10^{-1}\, h_2^3 h_1^6$$
$$+0.294\,755\,8406 \times 10^{-2}\, h_2^4 h_1^5 + 0.955\,965\,760\,5 \times 10^{-3}\, h_2^5 h_1^4$$
$$-0.587\,528\,210\,8E \times 10^{-4}\, h_2^6 h_1^3 + 0.187\,461\,3934 \times 10^{-5}\, h_1^2 h_2^7$$
$$-0.327\,893\,713 \times 10^{-7}\, h_1 h_2^8 + 0.738\,633\,806\,4 \times 10^{-10} h_2^9. \qquad (12)$$

The growth in magnitudes of the coefficients in (12) indicates that the series will diverge when h_1 and h_2 are greater than some minimum values. However, by re-summing using techniques such as Padé approximations, one can obtain meaningful results from this divergent series, as described in the next section.

4. Semiclassical energies of the double pendulum

To obtain quantum energies for the double pendulum, one faces the old problem of correctly quantizing it. The non-separability of velocities and coordinates in the Lagrangian (1) and momenta and coordinates in the Hamiltonian (2) shows the difficulty in writing the Schrödinger equation for the system. One may treat the double pendulum as a constrained system [12] and try to quantize it as such. However, the operator-ordering arbitrariness seems to persist and there is no assurance that the set of operator-ordered Hamiltonians will include the correct quantum Hamiltonian of the double pendulum. With our current knowledge of quantization, there always exists an element of faith in going from a classical to a quantum formalism for a dynamical system. Empirical support is needed for the validity of a quantizing procedure. In this work, we present semiclassical results which can be useful in efforts to arrive at the correct quantization of the double pendulum.

To find the semiclassical energies, we diagonalize the normal-form series for K in the basis set of two harmonic oscillators described by the Hamiltonians h_1 and h_2. The non-resonant case is very simple since the Hamiltonian K is a polynomial in h_1 and h_2 and hence diagonal in this basis set. For computing semiclassical energies, we use torus quantization [13] by replacing h_1, h_2 and their powers with $\hbar\omega_1\left(n_1 + \frac{1}{2}\right)$, $\hbar\omega_2(n_2 + \frac{1}{2})$ and their corresponding powers accordingly (note that the scalings of $\hbar$ and ω are as given in section 2). Here n_1 and n_2 are non-negative integers. For arbitrary l and m the results through K_1 can be obtained from (10). This quantization obviously excludes the non-commutativity of products of higher powers of the relevant quantum operators that one would expect to occur in an exact scheme of quantization. The quantum analogues [7, 11] of the classical Birkhoff–Gustavson normal form take account of operator ordering. However, these quantum analogues presume the existence of coordinate systems in which the original Hamiltonians are exactly quantizable. For the double pendulum, the authors are currently unaware of the existence of any such coordinate system.

The resonant case is more complicated [7] because K cannot be expressed as a polynomial in h_1 and h_2. In the presence of resonance, the matrix for K in the basis set of the two harmonic oscillators will have off-diagonal elements which should be evaluated (after using some operator ordering).

In general, the Birkhoff–Gustavson normal-form series has one of the following characteristics [9, 10]. (i) It has small radii of convergence, (ii) it is divergent, or (iii) it is asymptotic. These limitations should not dissuade us from using the normal-form approach. By invoking the techniques of analytic continuation and/or re-summation of divergent and asymptotic series, one usually obtains valuable results from the normal-form technique. The series for the double pendulum is such a case, and so requires re-summation to avoid divergencies in regions where it is able to provide meaningful results. To illustrate

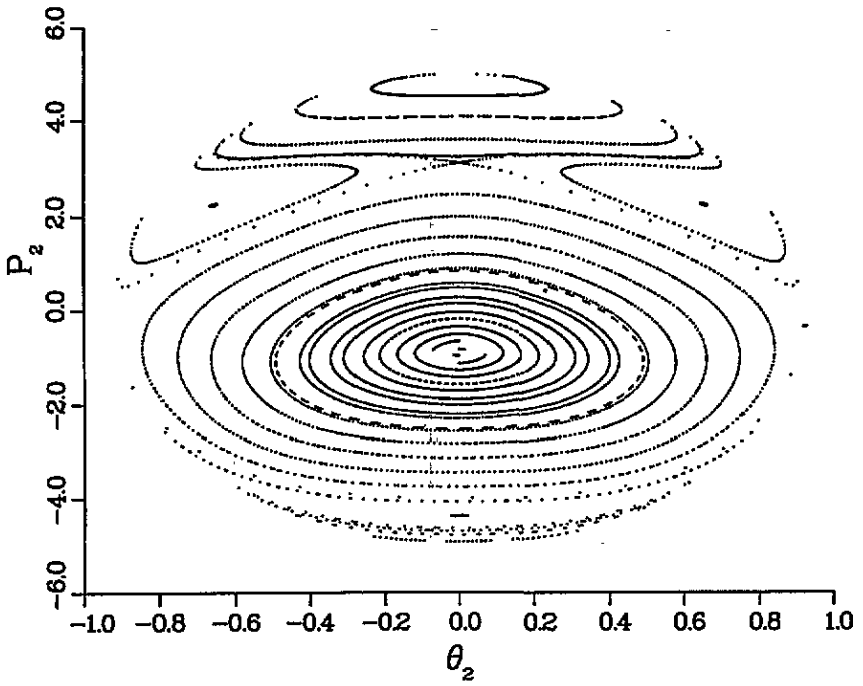

Figure 2. The Poincaré surface of section of the double pendulum for $\theta_1 = 0$, $\dot{\theta}_1 > 0$, $E = 2$, $m = 4$, $l = 1.25$, $\gamma = 1$.

this, we consider the particular case $m = 4$, $l = \frac{5}{4}$, $\gamma = 1$. For this example, it is clear from the surface of section (figure 2), that there is no large-scale chaos at energy $E = 2$. Our numerical results show that there is no such chaos for energies in the range $0 \leqslant E \approx 2$. In these situations, the normal-form analysis provides useful information about the system. One of the problems with convergence of the present normal form lies in the divergence of the series for H in θ_1 and θ_2 as mentioned in section 2. The other convergence difficulty lies in the fact that the system is chaotic at larger energies (see also the comments in [10] on the convergence of the Birkhoff–Gustavson normal form for an integrable system) and should not have a second constant of motion at these energies. Figure 3 shows large-scale chaos at energy $E = 7$. The existence of a normal-form series that converges and yields the values of H for all energies would imply integrability and hence no chaos.

To re-sum our normal-form series, we have tried several multivariate Padé summation techniques [14]. The two-variate diagonal $[M, M]$ Padé approximant scheme of Chisholm [15] has been the most satisfactory of all the algorithms that we have tried. The diagonal Padé approximants in h_1, h_2 of the series for K (equation (12)) were computed. For the [5, 5] approximant, we also computed the K_9 term. This [5, 5] approximant is given in the appendix. The semiclassical energies were obtained by using in these approximants the torus quantization discussed above and the results are given in table 1. For the entries in table 1, we have chosen $\hbar$ such that $\hbar\omega_1 = 1$. It is obvious that for smaller $\hbar$, there will be a larger number of energy levels below a given total energy for the system. It is also clear that for given n_1, n_2, the normal-form series and its Padé approximants will have better convergence for smaller $\hbar$. A full quantum calculation could be carried out for a given value of $\hbar$ and the ensuing results compared with those obtained through the normal-form approach.

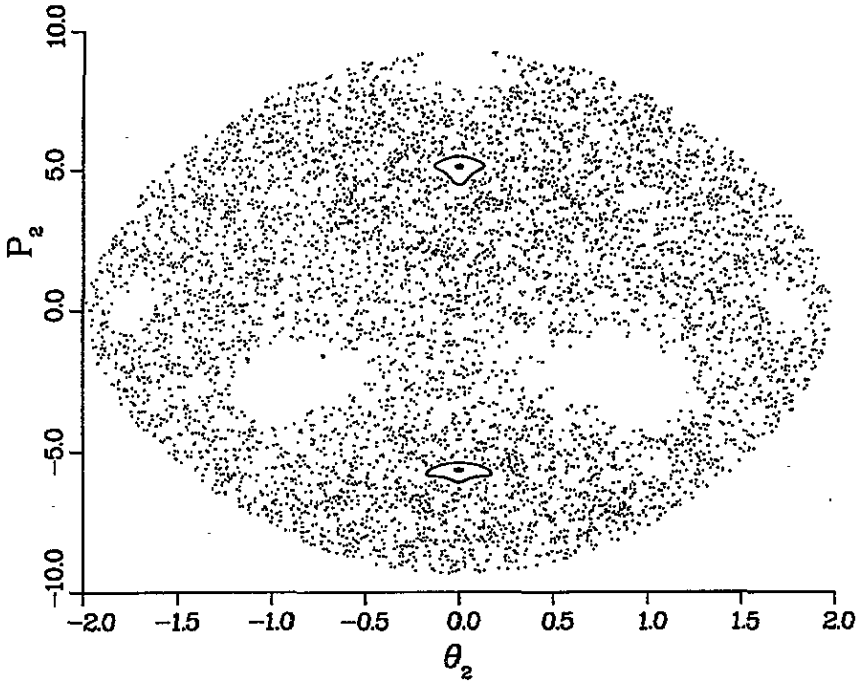

Figure 3. The Poincaré surface of section of the double pendulum for $\theta_1 = 0$, $\dot{\theta}_1 > 0$, $E = 7$, $m = 4$, $l = 1.25$, $\gamma = 1$.

Table 1. Semiclassical energies of the double pendulum for $m = 4$, $l = \frac{5}{4}$, $\gamma = 1$. The entries are two-variate $[M, M]$ diagonal Padé approximants of the normal-form series for K in h_1 and h_2. The numbers have been computed with $\hbar\omega_1 = 1$, $h_1 = n_1 + \frac{1}{2}$ $h_2 = (\omega_2/\omega_1)(n_2 + \frac{1}{2})$.

n_1, n_2	[1, 1]	[2, 2]	[3, 3]	[4, 4]	[5, 5]
0, 0	0.530 093 303 0	0.548 188 268 1	0.549 290 601 3	0.549 248 965 7	0.549 249 268 0
0, 1	0.766 302 356 7	0.784 207 309 9	0.785 111 737 0	0.785 196 735 7	0.785 197 689 7
0, 2	1.002 184 445	1.019 414 391	1.020 254 049	1.020 314 648	1.020 315 819
1, 0	1.033 534 622	1.192 472 329	1.224 456 722	1.223 986 527	1.224 015 952
1, 1	1.273 610 569	1.433 610 703	1.457 494 387	1.461 692 544	1.461 752 166
1, 2	1.513 806 018	1.673 788 253	1.696 163 575	1.698 660 465	1.698 715 051
2, 0	1.329 016 314	1.673 763 609	1.781 410 083	1.781 905 446	1.782 055 224
2, 1	1.571 807 212	1.920 273 530	1.996 805 542	2.013 546 806	2.013 803 465
2, 2	1.814 987 951	2.165 780 637	2.237 633 851	2.247 043 987	2.247 257 231

From table 1, we see that the Padé approximants converge well. Are these convergent Padé results in fact the semiclassical energies of the double pendulum? We believe that they are. Our support for this conviction is based on the following consideration. We numerically solved the equations of motion in the original polar coordinates for the energies quoted in table 1 and monitored the values of K and H as the system evolved. We observed that the values of the Padé re-summation for K always remained close to those of H, while the direct sum (12) diverged for most values of the energies considered. This implies that the Padé sum approximates H, while the direct sum fails to do so.

5. Approximate additional constants of motion

The normal-form approach can be used to generate approximate constants of motion independent of the Hamiltonian. Discussions of these integrals of motion for resonant and non-resonant cases are available in the literature; see for example Gustavson [3] and Kaluža and Robnik [9]. For a two-degree of freedom system, the normal form K is an integrable Hamiltonian by construction. With regard to our non-resonant K, both h_1 and h_2 are constants of motion, while K_0 can be used as a constant of motion for both the resonant and non-resonant cases. Let $I^\nu = f_0^\nu$ be a constant of motion in the normal-form coordinates (X, Z). We can transform I^ν to its image $I_\nu = f_\nu^0$ in the 'old' coordinates $(\mathcal{X}, \mathcal{Z})$. This inverse transformation can be carried out recursively by replacing (X, Z) with $(\mathcal{X}, \mathcal{Z})$ and using the following equation [16] instead of (7):

$$f_n^k = f_{n-1}^{k+1} - \sum_{m=0}^{n-1} \frac{(n-1)!}{m!\,(n-m-1)!} L_{m+1} f_{n-m-1}^k \qquad n \geqslant 1 \qquad k \geqslant 0 \tag{13}$$

where $L_j f$ is defined as in (7). By employing this procedure we obtain a perturbation series for I_ν and, as is common with perturbation methods, the issue of its convergence needs to be addressed. Generally speaking, perturbation series so obtained invariably require appropriate re-summation to enlarge their domains of application. In section 4, we have used a two-variate Padé approximation (in h_1 and h_2) to re-sum the normal-form series for K. For the double pendulum, I_ν is a function of four phase-space variables and a four-variate Padé approximation, amongst other summation techniques, can be tried. Our main purpose here is not to report on the optimal summation method for the function under consideration, but to promote the spirit of re-summation to extend the applicability of the results obtained from the normal-form approach. To this end, we consider a very simple-minded but general method. We write I_ν as a perturbation series in the parameter ϵ:

$$I_\nu = \sum_{m=0} \epsilon^m t_m \tag{14}$$

where the coefficients of expansion, the t_m, are the various perturbation terms independent of ϵ. We now work out a suitable $[M, N]$ Padé approximation in the single variable ϵ, and then finally set $\epsilon = 1$. It is clear that this technique is applicable to Hamiltonians of arbitrary degrees of freedom, both resonant and non-resonant cases. Given the simplicity of the method, the technique often works quite well and is worth trying.

As an application of this re-summation technique, we have transformed the non-resonant constant of motion h_1 to a function of the $(\mathcal{X}, \mathcal{Z})$ coordinates and then to another function of the coordinates θ_1, θ_2, P_1 and P_2 by inverting (5). The t_m are homogeneous polynomials of order $2(m+1)$. We have generated terms up to t_5 and have compared the results of the direct sum of (14) and its [2, 3] Padé approximation in figures 4 and 5. The t_0 and t_1 terms at the surface of section ($\theta_1 = 0$ and θ_1 increasing) are given by

$$t_0 = 1.094\,956\,582\theta_2^2 + 0.479\,464\,259\,0P_1^2 - 0.846\,525\,044\,5P_1P_2 + 0.373\,648\,628\,4P_2^2$$

$$\begin{aligned} t_1 = {} & 0.057\,496\,661\,90P_2^4 - 1.236\,708\,871\theta_2^2P_2^2 + 0.460\,767\,719\,0P_1^2P_2^2 \\ & -0.266\,900\,714\,8P_1P_2^3 + 0.099\,085\,853\,49P_1^4 + 0.372\,807\,083\,4\theta_2^4 \\ & +2.742\,758\,211\theta_2^2P_1P_2 - 0.350\,510\,959\,3P_1^3P_2 - 1.502\,291\,292\theta_2^2P_1^2. \end{aligned}$$

The condition of increasing θ_1 at the surface of section is satisfied by the inequality $P_1 - 0.8\,P_2\cos(\theta_2) > 0$, while P_1 is obtained in terms of θ_2 and P_2 from the equation $H = E$. A comparison of figures 2, 4 and 5 clearly shows that the Padé sum, as compared with the direct sum, is valid for much wider areas of the surface of section. It should be

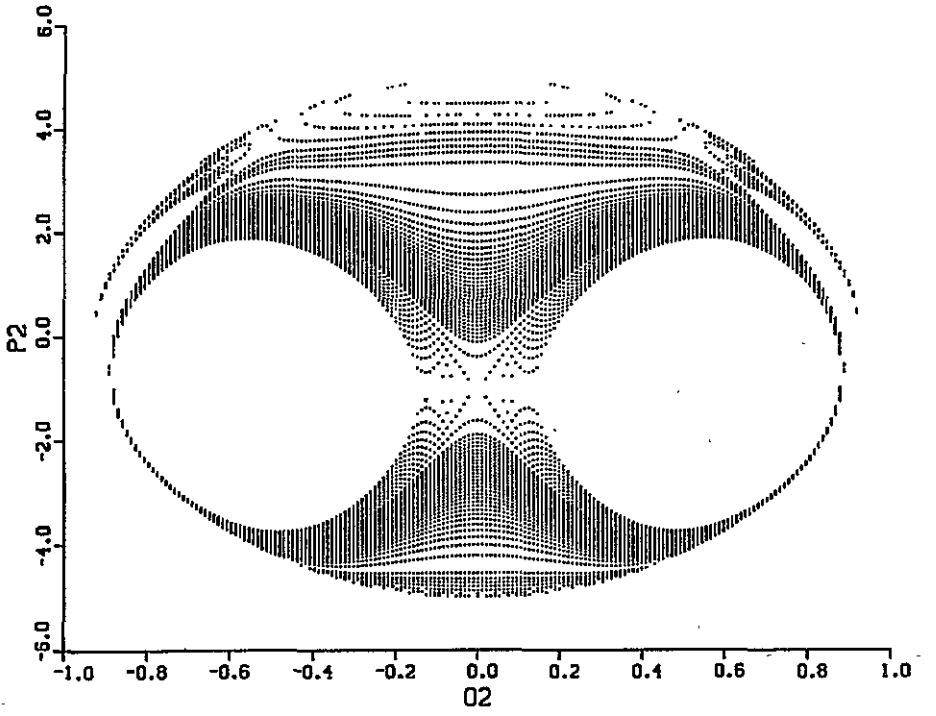

Figure 4. The level curves on the surface of section at $E = 2$ for the approximate constant of motion $I_1 = h_1$, obtained by directly summing the series (14), through t_5 ($\epsilon = 1$). Only the top part of the figure reproduces the exact results presented in figure 2. The curves were generated for values of the direct sum in the range 0–150.

pointed out that although there is no large-scale chaos at $E = 2$, one finds locally chaotic regions in finer analysis, implying that there are broken tori in the exact case. One such region is the neighbourhood of the hyperbolic point $\theta_2 = 0.0$, $P_2 = 3.12$ in figure 2.

6. Summary

We have presented a Birkhoff–Gustavson normal-form analysis of the double pendulum for a better understanding of this familiar system. We have discussed the convergence problem of this normal-form series and have shown that Padé summation provides meaningful results for the energy. Using torus quantization and a two-variate diagonal Padé summation technique, we have then presented semiclassical energies for a particular non-resonant case. These energies provide a basis of comparison for attempts at properly quantizing this system. By computing an approximate constant of motion using a different Padé summation technique, we have illustrated that efforts expended to re-sum normal-form results extend their domains of application.

Acknowledgment

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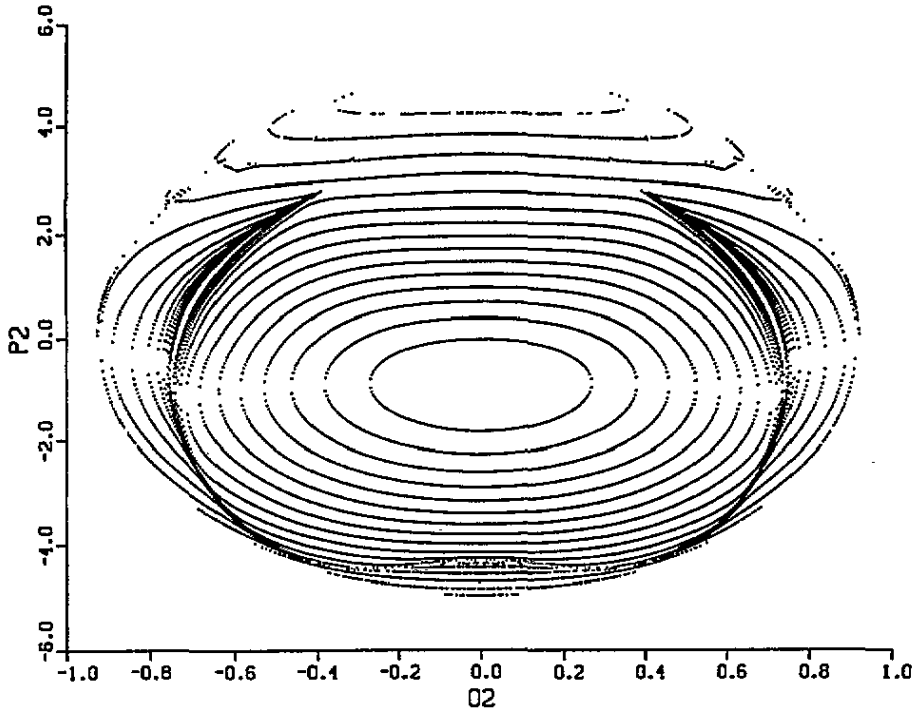

Figure 5. The level curves on the surface of section at $E = 2$ for the approximate constant of motion $I_1 = h_1$, obtained from [2, 3] Padé approximation of (14). The curves were generated for values of the approximant lying in the range 0–4. The Padé approximant clearly reproduces the exact results of figure 2 to a much better degree.

Appendix

The [5, 5] Padé approximant [5] is given by P/Q, where

$$\begin{aligned}P = {} & 0.787\,946\,051\,650\,282\,4 \times 10^{1}\, h_1 h_2 + 0.887\,983\,380\,952\,157\,3 \times 10^{3}\, h_2 h_1^2 \\ & +0.868\,093\,303\,152\,488\,4 \times 10^{3}\, h_2^2 h_1 + 0.459\,847\,312\,500\,641\,6 \times 10^{4}\, h_1^3 h_2 \\ & +0.488\,668\,228\,821\,135\,9 \times 10^{4}\, h_1^2 h_2^2 - 0.842\,786\,098\,280\,772\,1 \times 10^{2}\, h_2^3 h_1 \\ & +0.671\,350\,232\,191\,471\,8 \times 10^{4}\, h_2 h_1^4 - 0.101\,663\,577\,983\,330\,1 \times 10^{4}\, h_1^2 h_2^3 \\ & +0.749\,911\,027\,777\,748\,7 \times 10^{4}\, h_2^2 h_1^3 + 0.472\,182\,789\,352\,380\,7 \times 10^{0}\, h_2^4 h_1 \\ & +0.238\,849\,011\,614\,001\,6 \times 10^{4}\, h_2 h_1^5 + 0.134\,353\,894\,754\,752\,4 \times 10^{5}\, h_2^4 h_1^2 \\ & +0.211\,859\,513\,130\,413\,4 \times 10^{4}\, h_1^4 h_2^2 + 0.108\,249\,850\,066\,533\,1 \times 10^{5}\, h_1^3 h_2^3 \\ & +0.372\,927\,597\,022\,766\,1 \times 10^{0}\, h_2^5 h_1 + 0.438\,754\,245\,297\,878\,2 \times 10^{5}\, h_2^3 h_1^4 \\ & -0.734\,965\,356\,701\,506\,2 \times 10^{3}\, h_2^2 h_1^5 + 0.517\,339\,852\,355\,581\,5 \times 10^{5}\, h_2^4 h_1^3 \\ & +0.167\,980\,710\,330\,101\,8 \times 10^{2}\, h_2^5 h_1^2 + 0.462\,835\,329\,416\,602\,2 \times 10^{5}\, h_2^4 h_1^4 \\ & +0.290\,273\,191\,633\,141 \times 10^{5}\, h_2^3 h_1^5 + 0.110\,163\,023\,431\,103\,2 \times 10^{4}\, h_1^3 h_2^5 \\ & +0.354\,893\,920\,508\,162 \times 10^{4}\, h_2^5 h_1^4 + 0.785\,126\,025\,772\,407\,5 \times 10^{4}\, h_2^4 h_1^5 \\ & +0.790\,324\,153\,533\,543\,7 \times 10^{3}\, h_1^5 h_2^5 + 0.213\,808\,155\,987\,317\,9 \times 10^{0}\, h_2^2 \\ & +0.170\,386\,083\,297\,228\,2 \times 10^{2}\, h_1^3 + 0.721\,296\,005\,282\,319\,9 \times 10^{1}\, h_1^2\end{aligned}$$

$$
\begin{aligned}
&-0.298\,207\,786\,404\,205\,2\times10^{-1}\,h_2^3-0.928\,064\,030\,070\,438\,8\times10^{-3}\,h_2^4\\
&+0.146\,386\,082\,151\,427\,5\times10^{2}\,h_1^4+0.132\,483\,352\,034\,391\,7\times10^{-3}\,h_2^5\\
&+0.335\,845\,446\,587\,199\,2\times10^{1}\,h_1^5+h_1+h_2\\
Q=1&+0.868\,167\,580\,363\,627\,8\times10^{3}\,h_1h_2+0.494\,962\,527\,574\,352\,8\times10^{4}\,h_2h_1^2\\
&-0.788\,445\,313\,647\,144\,6\times10^{2}\,h_2^2h_1+0.834\,774\,293\,738\,613\,5\times10^{4}\,h_1^3h_2\\
&-0.982\,375\,252\,210\,205\,5\times10^{3}\,h_1^2h_2^2+0.271\,707\,289\,779\,347\,6\times10^{-1}\,h_2^3h_1\\
&+0.409\,017\,163\,808\,099\times10^{4}\,h_2h_1^4+0.134\,289\,932\,444\,374\,8\times10^{5}\,h_1^2h_2^3\\
&-0.252\,174\,933\,494\,380\,5\times10^{4}\,h_2^2h_1^3+0.367\,779\,330\,301\,124\,7\times10^{0}\,h_2^4h_1\\
&+0.215\,534\,619\,502\,923\,1\times10^{3}\,h_2h_1^5+0.100\,737\,977\,954\,382\,1\times10^{3}\,h_2^4h_1^2\\
&-0.164\,854\,536\,462\,610\,7\times10^{4}\,h_1^4h_2^2+0.513\,681\,362\,135\,296\,9\times10^{5}\,h_1^3h_2^3\\
&+0.238\,387\,267\,928\,842\,6\times10^{-2}\,h_2^5h_1+0.437\,791\,755\,695\,332\,4\times10^{5}\,h_2^3h_1^4\\
&-0.113\,613\,975\,764\,563\,4\times10^{3}\,h_2^2h_1^5+0.144\,833\,644\,039\,387\,4\times10^{4}\,h_2^4h_1^3\\
&+0.138\,493\,967\,300\,077\,5\times10^{1}\,h_2^5h_1^2+0.389\,967\,734\,747\,687\,4\times10^{4}\,h_2^4h_1^4\\
&+0.367\,098\,618\,589\,930\,3\times10^{4}\,h_2^3h_1^5+0.123\,312\,488\,192\,926\,7\times10^{2}\,h_1^3h_2^5\\
&+0.315\,877\,334\,918\,419\,2\times10^{2}\,h_2^5h_1^4+0.775\,411\,341\,291\,540\,2\times10^{3}\,h_2^4h_1^5\\
&+0.301\,663\,890\,294\,410\,8\times10^{2}\,h_1^5h_2^5+0.764\,093\,448\,624\,023\,8\times10^{1}\,h_1\\
&+0.220\,064\,491\,801\,048\,1\times10^{0}\,h_2-0.283\,881\,824\,595\,439\,2\times10^{-1}\,h_2^2\\
&+0.198\,739\,766\,336\,943\,7\times10^{2}\,h_1^3+0.197\,587\,685\,740\,593\,6\times10^{2}\,h_1^2\\
&-0.109\,444\,465\,828\,329\,8\times10^{-2}\,h_2^3+0.123\,816\,735\,715\,949\,6\times10^{-3}\,h_2^4\\
&+0.641\,978\,644\,274\,061\,6\times10^{1}\,h_1^4+0.701\,558\,930\,039\,153\,6\times10^{-6}\,h_2^5\\
&+0.232\,687\,782\,645\,944\,7\times10^{0}\,h_1^5
\end{aligned}
$$

References

[1] Ford J and Mantica G 1992 *Am. J. Phys.* **60** 1086
Ford J and Ilg M 1992 *Phys. Rev.* A **45** 6165
Ford J, Mantica G and Ristow G H 1991 *Physica* **50D** 493

[2] Shinbrot T, Grebogi C, Wisdom J and Yorke J A 1992 *Am. J. Phys.* **60** 491
Richter P H and Scholz H-J 1984 *Stochastic Phenomena and Chaotic Behavior in Complex Systems* ed P Schuster (Berlin: Springer)

[3] Gustavson F G 1966 *Astron. J.* **71** 670
Birkhoff G D 1966 *Dynamical Systems* (Providence, RI: American Mathematical Society)

[4] Jaffé C and Reinhardt W P 1982 *J. Chem. Phys.* **77** 5192
Shirts R B and Reinhardt W P 1982 *J. Chem. Phys.* **77** 5204
Swimm R T and Delos J B 1979 *J. Chem. Phys.* **71** 1706
Noid D W and Marcus R A 1977 *J. Chem. Phys.* **67** 559
Noid D W, Koszykowski M L, Tabor M and Marcus R A 1980 *J. Chem. Phys.* **73** 6169
Churchill R C, Pecelli G and Rod D L 1978 *Lecture Notes in Physics* vol 93 (Berlin: Springer)

[5] Note that all algebraic and numerical computations in this work have been performed using MAPLE. All the numerical values quoted here were actually computed to 30 digits of accuracy

[6] Nayfeh A H 1973 *Perturbation Methods* (New York: Wiley)
McRae S M and Vrscay E R 1992 *J. Math. Phys.* **33** 3004

[7] Ali M K 1985 *J. Math. Phys.* **26** 2565 (Note that equation (63) needs to be corrected by the addition of the expression $-(9\,b^2\hbar^2/4\,\omega)\,[a\,a\dagger\,a\,a\dagger+a\dagger\,a\,a\dagger\,a]$.).

[8] Alvarez G, Graffi S and Silverstone H J 1988 *Phys. Rev.* A **38** 1687
Alvarez G 1989 *J. Phys. A: Math. Gen.* **22** 617

[9] Kaluža M and Robnik M 1992 *J. Phys. A: Math. Gen.* **25** 5311

[10] Ali M K, Wood W R and Devitt J S 1986 *J. Math. Phys.* **27** 180
Ali M K and Wood W R 1989 *J. Math. Phys.* **30** 1238
[11] Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 109
Uzer T and Marcus R A 1984 *J. Chem. Phys.* **81** 5013
Eckhardt B 1986 *J. Phys. A: Math. Gen.* **19** 2961
Crehan P 1990 *J. Phys. A: Math. Gen.* **23** 5815
Kummer M 1993 private communication
[12] Huber H and Büttner H 1992 *J. Phys. A: Math. Gen.* **25** L1111
Blau S K 1991 *Ann. Phys., NY* **205** 392
Lemos N A 1990 *J. Math. Phys.* **31** 348
Henneaux M 1985 *Phys. Rep.* **126** 1
Dirac P A M 1967 *Lectures on Quantum Mechanics* (New York: Academic)
[13] Einstein A 1917 *Verh. Deutsch. Phys. Ges. Berlin* **19** 82
Brillouin L 1926 *J. Phys. Radium* **7** 353
Keller J B 1958 *Ann. Phys., NY* **4** 180
[14] Werner H 1986 *Numer. Math.* **48** 429
Cuyt A 1986 *BIT* **26** 71
Cuyt A and Verdonk B 1984 *Numer. Math.* **43** 293
Jones R H 1976 *J. Approx. Theory* **16** 201
Watson P J 1974 *J. Phys. A: Math. Gen.* **7** L167
[15] Chisholm J S 1973 *Math. Comput.* **27** 841
[16] Kamel A A 1969 *Celes. Mech.* **1** 190